

THE PRIME NUMBER RACE AND ZEROS OF DIRICHLET L -FUNCTIONS OFF THE CRITICAL LINE. III

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ABSTRACT. We show, for any $q \geq 3$ and distinct reduced residues $a, b \pmod{q}$, the existence of certain hypothetical sets of zeros of Dirichlet L -functions lying off the critical line implies that $\pi(x; q, a) < \pi(x; q, b)$ for a set of real x of asymptotic density 1.

1 Introduction

For $(a, q) = 1$, let $\pi(x; q, a)$ denote the number of primes $p \leq x$ with $p \equiv a \pmod{q}$. The study of the relative magnitudes of the functions $\pi(x; q, a)$ for a fixed q and varying a is known colloquially as the “prime race problem” or “Shanks-Rényi prime race problem”. For a survey of problems and results on prime races, the reader may consult the papers [4] and [5]. One basic problem is the study of $P_{q;a_1, \dots, a_r}$, the set of real numbers $x \geq 2$ such that $\pi(x; q, a_1) > \dots > \pi(x; q, a_r)$. It is generally believed that all sets $P_{q;a_1, \dots, a_r}$ are unbounded. Assuming the Generalized Riemann Hypothesis for Dirichlet L -functions modulo q (GRH_q) and that the nonnegative imaginary parts of zeros of these L -functions are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown for any r -tuple of reduced residue classes a_1, \dots, a_r modulo q , that $P_{q;a_1, \dots, a_r}$ has a positive logarithmic density (although the density may be quite small in some cases).

In [2] and [3], Ford and Konyagin investigated how possible violations of the Generalized Riemann Hypothesis (GRH) would affect prime number races. In [2], they proved that the existence of certain sets of zeros off the critical line would imply that some of the sets $P_{q;a_1, a_2, a_3}$ are bounded, giving a negative answer to the prime race problem with $r = 3$. Paper [3] was devoted to similar questions for r -way prime races with $r > 3$. One result from [3] states that for any q , $r \leq \phi(q)$ and set $\{a_1, \dots, a_r\}$ of reduced residues modulo q , the existence of certain hypothetical sets of zeros of Dirichlet L -functions modulo q implies that at most $r(r-1)$ of the sets $P_{q;\sigma(a_1), \dots, \sigma(a_r)}$ are unbounded, σ running over all permutations of $\{a_1, \dots, a_r\}$.

In this paper, we investigate the effect of zeros of L -functions lying off the critical line for two way prime races. This case is harder, since it is unconditionally proved that for certain races $\{q; a, b\}$ the set $P_{q;a, b}$ is unbounded. For example, Littlewood [11] proved that $P_{4;3,1}$, $P_{4;1,3}$, $P_{3;1,2}$ and $P_{3;2,1}$ are unbounded. Later Knapowski and Turàn ([9], [10]) proved for many q, a, b that $\pi(x; q, b) - \pi(x; q, a)$ changes sign infinitely often and more recently Sneed [13] showed that $P_{q;a, b}$ is unbounded for every $q \leq 100$ and all possible pairs (a, b) .

Nevertheless, we prove that the existence of certain zeros off the critical line would imply that the set $P_{q;a, b}$ has asymptotic density zero, in contrast with a conditional result of Kaczorowski [7] on GRH, which asserts that $P_{q;1, b}$ and $P_{q; b, 1}$ have positive lower densities for all $(b, q) = 1$.

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Let $q \geq 3$ be a positive integer and a, b be distinct reduced residues modulo q . Moreover, for any set \mathcal{S} of real numbers we define $\mathcal{S}(X) = \mathcal{S} \cap [2, X]$.

Theorem 1.1. *Let $q \geq 3$ and suppose that a and b are distinct reduced residues modulo q . Let χ be a nonprincipal Dirichlet character with $\chi(a) \neq \chi(b)$, and put $\xi = \arg(\chi(a) - \chi(b)) \in [0, 2\pi)$. Suppose $\frac{1}{2} < \sigma < 1$, $0 < \delta < \sigma - \frac{1}{2}$, $A > 0$, and $\mathcal{B} = \mathcal{B}(\xi, \sigma, \delta, A)$ is a multiset of complex numbers satisfying the conditions listed in Section 2. If $L(\rho, \chi) = 0$ for all $\rho \in \mathcal{B}$, $L(s, \chi)$ has no other zeros in the region $\{s : \operatorname{Re}(s) \geq \sigma - \delta, \operatorname{Im}(s) \geq 0\}$, and for all other nonprincipal characters χ' modulo q , $L(s, \chi') \neq 0$ in the region $\{s : \operatorname{Re}(s) \geq \sigma - \delta, \operatorname{Im}(s) \geq 0\}$, then*

$$\lim_{X \rightarrow \infty} \frac{\operatorname{meas}(P_{q;a,b}(X))}{X} = 0.$$

Remarks. Such χ exists whenever a and b are distinct modulo q . The sets \mathcal{B} have the property that any $\rho \in \mathcal{B}$ has real part in $[\sigma - \delta, \sigma]$, imaginary part greater than A , and multiplicity $O((\log \operatorname{Im}(\rho))^{3/4})$ (that is, the multiplicities are much smaller than known bounds on the multiplicity of zeros of Dirichlet L -functions). The number of elements of \mathcal{B} (counted with multiplicity) with imaginary part less than T is $O((\log T)^{5/4})$, and thus \mathcal{B} is quite a “thin” set. Also, we note that if $L(\beta + i\gamma, \chi) = 0$ then $L(\beta - i\gamma, \overline{\chi}) = 0$, which is a consequence of the functional equation for Dirichlet L -functions (See e.g. Ch. 9 of [1]). The point of Theorem 1.1 is that proving

$$\limsup_{X \rightarrow \infty} \frac{\operatorname{meas}(P_{q;a,b}(X))}{X} > 0$$

requires showing that the multiset of zeros of $L(s, \chi)$ cannot contain any of the multisets \mathcal{B} . This is beyond what is possible with existing technology (see e.g. [6] for the best known estimates for multiplicities of zeros).

Our method works as well for the difference $\pi(x) - \operatorname{li}(x)$, the error term in the prime number theorem. Littlewood [11] established that this quantity changes sign infinitely often. Let P_1 be the set of real numbers $x \geq 2$ such that $\pi(x) > \operatorname{li}(x)$. In [8] Kaczorowski proved, assuming the Riemann Hypothesis, that both P_1 and \overline{P}_1 have positive lower densities. Assuming the Riemann Hypothesis and that the nonnegative imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$ are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown that P_1 has a positive logarithmic density $\delta_1 \approx 0.00000026$. In contrast with these results we prove that the existence of certain zeros of $\zeta(s)$ off the critical line would imply that the set P_1 has asymptotic density zero (or asymptotic density 1).

Theorem 1.2. *Suppose $\frac{1}{2} < \sigma < 1$, $0 < \delta < \sigma - \frac{1}{2}$ and $A > 0$. (i) If $\xi = 0$, $\mathcal{B} = \mathcal{B}(\xi, \sigma, \delta, A)$ satisfies the conditions of Section 2, $\zeta(\rho) = 0$ for all $\rho \in \mathcal{B}$, and $\zeta(s)$ has no other zeros in the region $\{s : \operatorname{Re}(s) \geq \sigma - \delta, \operatorname{Im}(s) \geq 0\}$, then*

$$\lim_{X \rightarrow \infty} \frac{\operatorname{meas}(P_1(X))}{X} = 0.$$

(ii) If $\xi = \pi$, \mathcal{B} satisfies the conditions of Section 2, $\zeta(\rho) = 0$ for all $\rho \in \mathcal{B}$, and $\zeta(s)$ has no other zeros in the region $\{s : \operatorname{Re}(s) \geq \sigma - \delta, \operatorname{Im}(s) \geq 0\}$, then

$$\lim_{X \rightarrow \infty} \frac{\operatorname{meas}(P_1(X))}{X} = 1.$$

We omit the proof of Theorem 1.2, as it is nearly identical to the proof of Theorem 1.1 in the case $q = 4$.

2 The construction of \mathcal{B}

For $j \geq 1$, suppose that

$$(2.1) \quad \begin{aligned} \exp(j^8) &\leq \gamma_j \leq 2 \exp(j^8), \quad \left| \delta_j - \frac{1}{j^8} \right| \leq \frac{1}{j^9}, \\ \text{and} \quad \left| \theta_j - \frac{\xi - \pi/2}{j^{16}} \right| &\leq \frac{1}{j^{17}}. \end{aligned}$$

We choose j_0 so large that for all $j \geq j_0$, $\gamma_j > A$ and $\sigma - \delta \leq \sigma - \delta_j$. Then we take \mathcal{B} to be the union, over $j \geq j_0$ and $1 \leq k \leq j^3$, of $m(k, j) = k(j^3 + 1 - k)$ copies of $\rho_{j,k}$, where

$$\rho_{j,k} = \sigma - \delta_j + i(k\gamma_j + \theta_j).$$

3 Preliminary Results

The following classical-type explicit formula was established in Lemma 1.1 of [2] when $x' = x$. The slightly more general result below, which is more convenient for us, is proved in exactly the same way.

Lemma 3.1. *Let $\beta \geq 1/2$ and for each non-principal character $\chi \bmod q$, let $B(\chi)$ be the sequence of zeros (duplicates allowed) of $L(s, \chi)$ with $\operatorname{Re}(s) > \beta$ and $\operatorname{Im}(s) > 0$. Suppose further that all $L(s, \chi)$ are zero-free on the real segment $\beta < s < 1$. If $(a, q) = (b, q) = 1$, x is sufficiently large and $x' \geq x$, then*

$$\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = -2\operatorname{Re} \left(\sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} (\overline{\chi}(a) - \overline{\chi}(b)) \sum_{\substack{\rho \in B(\chi) \\ |\operatorname{Im}(\rho)| \leq x'}} f(\rho) \right) + O(x^\beta \log^2 x),$$

where

$$f(\rho) := \frac{x^\rho}{\rho \log x} + \frac{1}{\rho} \int_2^x \frac{t^\rho}{t \log^2 t} dt = \frac{x^\rho}{\rho \log x} + O\left(\frac{x^{\operatorname{Re}(\rho)}}{|\rho|^2 \log^2 x}\right).$$

Remark. For Theorem 1.2, we use a similar explicit formula for $\pi(x)$ in terms of the zeros $B(\zeta)$ of the Riemann zeta function which satisfy $\Re \rho > \beta$ and $\Im \rho > 0$:

$$\pi(x) = \operatorname{li}(x) - 2\Re \sum_{\substack{\rho \in B(\zeta) \\ |\Im \rho| \leq x'}} f(\rho) + O(x^\beta \log^2 x).$$

Using properties of the Fejér kernel we prove the following key proposition.

Proposition 3.2. *Let $\gamma \geq 1$, $L \geq 4$ and $X \geq 2$. Define*

$$F_{\gamma,L}(x) = \sum_{k=1}^{L-1} (L-k) \cos(k\gamma \log x).$$

Then

$$\operatorname{meas} \left\{ x \in [1, X] : F_{\gamma,L}(x) \geq -\frac{L}{4} \right\} \ll \frac{X}{\sqrt{L}}.$$

Proof. The Fejér kernel satisfies the following identity

$$\frac{1}{L} \left(\frac{\sin\left(\frac{L\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \right)^2 = 1 + 2 \sum_{k=1}^{L-1} \left(1 - \frac{k}{L}\right) \cos(k\theta).$$

This yields

$$F_{\gamma,L}(x) = \frac{\sin^2\left(\frac{L\gamma \log x}{2}\right)}{2 \sin^2\left(\frac{\gamma \log x}{2}\right)} - \frac{L}{2}.$$

Therefore, if $F_{\gamma,L}(x) \geq -L/4$ then

$$\sin^2\left(\frac{\gamma \log x}{2}\right) \leq \frac{2}{L} \sin^2\left(\frac{L\gamma \log x}{2}\right) \leq \frac{2}{L}.$$

Hence,

$$\left\| \frac{\gamma \log x}{2\pi} \right\| \leq \varepsilon := \frac{1}{\sqrt{2L}},$$

where $\|t\|$ denotes the distance to the nearest integer. This implies

$$\begin{aligned} \text{meas} \left\{ x \in [1, X] : F_{\gamma,L}(x) \geq -\frac{L}{4} \right\} &\leq \text{meas} \left\{ x \in [1, X] : \left\| \frac{\gamma \log x}{2\pi} \right\| \leq \varepsilon \right\} \\ &\leq \sum_{0 \leq k \leq \frac{\gamma \log X}{2\pi} + \varepsilon} e^{2\pi(k+\varepsilon)/\gamma} - e^{2\pi(k-\varepsilon)/\gamma} \\ &\ll \frac{\varepsilon}{\gamma} \sum_{0 \leq k \leq \frac{\gamma \log X}{2\pi} + \varepsilon} e^{2\pi(k+\varepsilon)/\gamma} \ll \varepsilon X. \quad \square \end{aligned}$$

4 Proof of Theorem 1.1

Suppose X is large and $\sqrt{X} \leq x \leq X$. For brevity, let

$$\Delta = \phi(q)(\pi(x; q, a) - \pi(x; q, b)).$$

It follows from Lemma 3.1 with $x' = \max(x, \max\{j^3 \gamma_j : \gamma_j \leq x\})$ that

$$\begin{aligned} \Delta &= -\frac{2}{\log x} \text{Re} \left((\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\gamma_j \leq x} \sum_{k=1}^{j^3} \frac{x^{\sigma-\delta_j+i(k\gamma_j+\theta_j)} m(k, j)}{\sigma - \delta_j + i(k\gamma_j + \theta_j)} \right) \\ &\quad + O \left(\frac{x^\sigma}{\log^2 x} \sum_{\gamma_j \leq x} \frac{x^{-\delta_j}}{\gamma_j^2} \sum_{k=1}^{j^3} \frac{m(k, j)}{k^2} + x^{\sigma-\delta} \log^2 x \right) \\ (4.1) \quad &= \frac{2x^\sigma}{\log x} \text{Re} \left(i(\bar{\chi}(a) - \bar{\chi}(b)) \sum_{\gamma_j \leq x} \frac{x^{-\delta_j}}{\gamma_j} \sum_{k=1}^{j^3} x^{i(k\gamma_j+\theta_j)} (j^3 + 1 - k) \right) \\ &\quad + O \left(\frac{x^\sigma}{\log x} \sum_{\gamma_j \leq x} \frac{j^4 x^{-\delta_j}}{\gamma_j^2} + x^{\sigma-\delta} \log^2 x \right). \end{aligned}$$

Note that

$$\frac{x^{-\delta_j}}{\gamma_j} = \exp \left(-\frac{\log x}{j^8} \left(1 + O \left(\frac{1}{j} \right) \right) - j^8 + O(1) \right).$$

The maximum of this function over j occurs around $J = J(x) := \lceil (\log x)^{1/16} \rceil$. In this case we have $\log x = J^{16}(1 + O(1/J))$ so that

$$(4.2) \quad \frac{x^{-\delta_J}}{\gamma_J} = \exp \left(-2J^8 + O(J^7) \right) = \exp \left(-2(\log x)^{1/2} + O((\log x)^{7/16}) \right).$$

We will prove that most of the contribution to the main term on the right hand side of (4.1) comes for the j 's in the range $J - J^{3/4} \leq j \leq J + J^{3/4}$. First, if $j \geq 3J/2$ or $j \leq J/2$ then

$$\frac{x^{-\delta_j}}{\gamma_j} \ll \exp(-4J^8) \ll \exp(-(\log x)^{1/2}) \frac{x^{-\delta_J}}{\gamma_J}.$$

Now suppose that $J/2 < j < J - J^{3/4}$ or $J + J^{3/4} < j < 3J/2$. Write $j = J + r$ with $J^{3/4} < |r| < J/2$. For $x > 0$, $x + 1/x = 2 + (x - 1)^2/x$, hence

$$\left(1 + \frac{r}{J}\right)^8 + \left(1 + \frac{r}{J}\right)^{-8} \geq \left(1 + \left|\frac{r}{J}\right|\right)^8 + \left(1 + \left|\frac{r}{J}\right|\right)^{-8} \geq 2 + \frac{(8r/J)^2}{1 + 8r/J} \geq 2 + 12(r/J)^2.$$

We infer from (4.2) that

$$\begin{aligned} \frac{x^{-\delta_j}}{\gamma_j} &= \exp \left(-\frac{J^{16}}{j^8} \left(1 + O \left(\frac{1}{J} \right) \right) - j^8 \right) \\ &= \exp \left(-J^8 \left(\left(1 + \frac{r}{J}\right)^8 + \left(1 + \frac{r}{J}\right)^{-8} \right) + O(J^7) \right) \\ &\leq \exp \left(-2J^8 \left(1 + \frac{6}{\sqrt{J}} \right) + O(J^7) \right) \\ &\ll \exp \left(-2(\log x)^{1/3} \right) \frac{x^{-\delta_J}}{\gamma_J}. \end{aligned}$$

Since $\gamma_j \leq x$ implies that $j \ll (\log x)^{1/8}$, the contribution of the terms $1 \leq j < J - J^{3/4}$ or $J + J^{3/4} < j$ to the main term of (4.1) is

$$(4.3) \quad \ll \exp \left(-2(\log x)^{1/3} \right) \frac{x^{\sigma-\delta_J}}{\gamma_J} \sum_{j \leq (\log x)^{1/4}} \sum_{k=1}^{j^3} (j^3 + 1 - k) \ll \exp \left(-(\log x)^{1/3} \right) \frac{x^{\sigma-\delta_J}}{\gamma_J}.$$

Similarly, we have

$$\begin{aligned} \frac{x^{-\delta_j}}{\gamma_j^2} &= \exp \left(-\frac{\log x}{j^8} \left(1 + O \left(\frac{1}{j} \right) \right) - 2j^8 + O(1) \right) \\ &\ll \exp \left(-2\sqrt{2}(\log x)^{1/2}(1 + o(1)) \right) \\ &\ll \exp \left(-2(\log x)^{1/3} \right) \frac{x^{-\delta_J}}{\gamma_J}, \end{aligned}$$

which follows from (4.2) along with the fact that the maximum of $f(t) = -\log x/t^8 - 2t^8$ occurs at $t = (\log x/2)^{1/16}$. Hence, using (4.2), the contribution of the error term of (4.1) is

$$(4.4) \quad \ll \exp(-2(\log x)^{1/3}) \frac{x^{\sigma-\delta_J}}{\gamma_J} \sum_{j \leq (\log x)^{1/4}} j^4 + x^{\sigma-\delta} \log^2 x \ll \exp(-(\log x)^{1/3}) \frac{x^{\sigma-\delta_J}}{\gamma_J}.$$

Therefore, inserting the bounds (4.3) and (4.4) in (4.1) we deduce that

$$(4.5) \quad \Delta = \frac{2x^\sigma}{\log x} \operatorname{Re} \left(i(\overline{\chi}(a) - \overline{\chi}(b)) \sum_{|j-J| \leq J^{3/4}} \frac{x^{-\delta_j}}{\gamma_j} \sum_{k=1}^{j^3} \exp(i(k\gamma_j + \theta_j) \log x) (j^3 + 1 - k) \right) \\ + O \left(\exp(-(\log x)^{1/3}) \frac{x^{\sigma-\delta_J}}{\gamma_J} \right).$$

Let $J - J^{3/4} \leq j \leq J + J^{3/4}$. Then $j^{16} = J^{16} (1 + O(J^{-1/4}))$. Hence we get

$$\begin{aligned} \theta_j \log x &= \left(\arg(\chi(a) - \chi(b)) - \frac{\pi}{2} \right) \frac{\log x}{j^{16}} + O \left(\frac{\log x}{j^{17}} \right) \\ &= \left(\arg(\chi(a) - \chi(b)) - \frac{\pi}{2} \right) + O \left(\frac{1}{J^{1/4}} \right). \end{aligned}$$

This implies

$$i(\overline{\chi}(a) - \overline{\chi}(b)) \exp(i\theta_j \log x) = |\chi(a) - \chi(b)| \left(1 + O \left(\frac{1}{J^{1/4}} \right) \right),$$

since $e^{i \arg z} = z/|z|$. Inserting this estimate in (4.5) we obtain

$$(4.6) \quad \Delta = \left(1 + O \left(\frac{1}{\log^{1/64} x} \right) \right) 2|\chi(a) - \chi(b)| \sum_{|j-J| \leq J^{3/4}} \frac{x^{\sigma-\delta_j}}{\gamma_j \log x} F_{\gamma_j, j^3}(x) \\ + O \left(\exp(-(\log x)^{1/3}) \frac{x^{\sigma-\delta_J}}{\gamma_J} \right).$$

For $x \in [\sqrt{X}, X]$ we have $\frac{1}{4}(\log X)^{1/16} \leq J - J^{3/4}$ and $J + J^{3/4} \leq 4(\log X)^{1/16}$ if X is sufficiently large, since $J = (\log x)^{1/16} + O(1)$. We define

$$\Omega := \left\{ x \in [\sqrt{X}, X] : F_{\gamma_j, j^3}(x) \leq -\frac{j^3}{4} \text{ for all } \frac{1}{4}(\log X)^{1/16} \leq j \leq 4(\log X)^{1/16} \right\}.$$

Then it follows from Proposition 3.2 that

$$(4.7) \quad \begin{aligned} \operatorname{meas} \Omega &= X + O \left(X \sum_{\frac{1}{4}(\log X)^{1/16} \leq j \leq 4(\log X)^{1/16}} \frac{1}{j^{3/2}} + \sqrt{X} \right) \\ &= X (1 + O((\log X)^{-1/32})). \end{aligned}$$

Furthermore, if $x \in \Omega$ then we infer from (4.6) that

$$\begin{aligned} \Delta &\leq -\frac{1}{3}|\chi(a) - \chi(b)| \sum_{|j-J| \leq J^{3/4}} \frac{j^3 x^{\sigma-\delta_j}}{\gamma_j \log x} + O\left(\exp\left(-(\log x)^{1/3}\right) \frac{x^{\sigma-\delta_J}}{\gamma_J}\right). \\ &\leq -\frac{1}{3}|\chi(a) - \chi(b)| \frac{J^3 x^{\sigma-\delta_J}}{\gamma_J \log x} (1 + o(1)) < 0 \end{aligned}$$

if X is sufficiently large, which completes the proof.

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